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# A SIMPLE HEURISTIC APPROACH TO SIMPLEX EFFICIENCY

by
SHELDON M. ROSS

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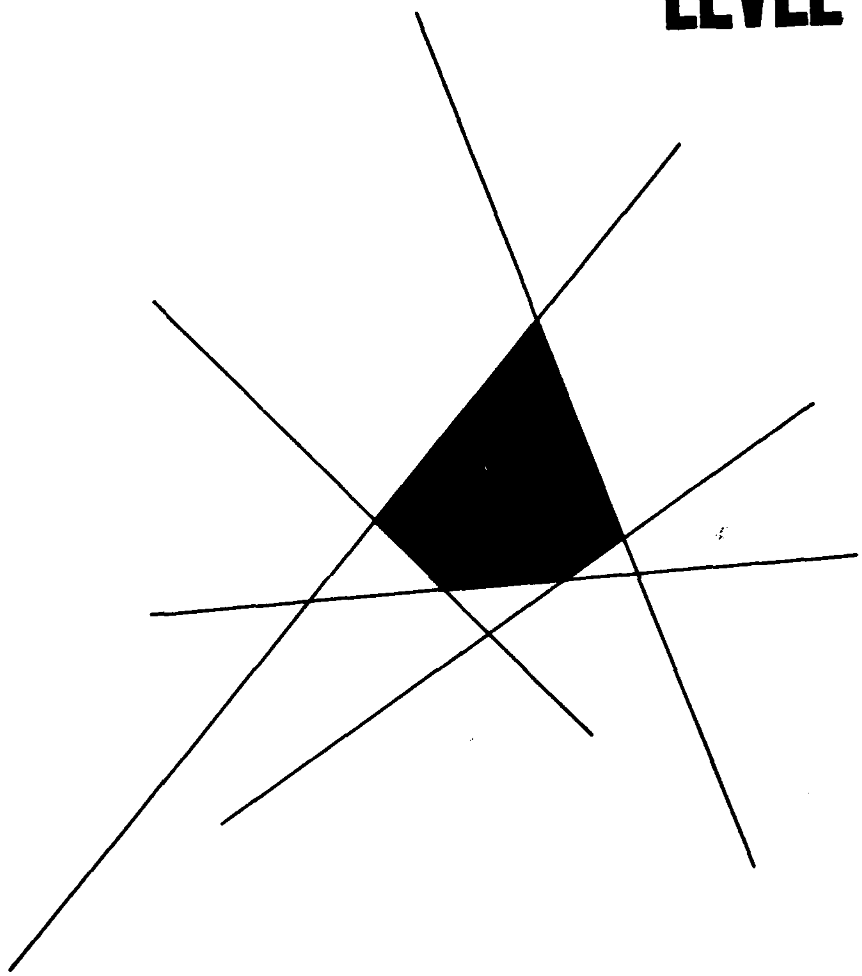

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by

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## ABSTRACT

Consider the standard linear program:

$$\text{Minimize } \underline{c}\,\underline{x}$$

$$\text{subject to: } A\,\underline{x} = \underline{b}$$

$$\underline{x} \geq 0$$

where A is an $m \times n$ matrix. The simplex algorithm solves this linear program by moving from extreme point of the feasibility region to a better (in terms of the objective function $\underline{c}\,\underline{x}$) extreme point (via the pivot operation) until the optimal is reached. In order to obtain a feel for the number of necessary iterations, we consider a simple probabilistic (Markov chain) model as to how the algorithm moves along the extreme points. At first we suppose that if at any time the algorithm is at the jth best extreme point then after the next pivot the resulting extreme point is equally likely to be any of the $j - 1$ best. Under this assumption, we show that the time to get from the Nth best to the best extreme point has approximately, for large N, a Poisson distribution with mean equal to the logarithm (base e) of N. We also consider a more general probabilistic model in which we drop the uniformity assumption and suppose that when at the jth best the next one is chosen probabilistically according to weights $w_i$, $i = 1, \ldots, j - 1$.

# A SIMPLE HEURISTIC APPROACH TO SIMPLEX EFFICIENCY

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## 1. INTRODUCTION

Consider the standard linear program:

$$\begin{aligned} &\text{Minimize } \underline{c}\ \underline{x} \\ &\text{subject to: } A\,\underline{x} = \underline{b} \\ &\qquad\qquad\qquad \underline{x} \geq 0 \end{aligned}$$

where $A$ is an $m \times n$ matrix. The simplex algorithm solves this linear program by moving from extreme point of the feasibility region to a better (in terms of the objective function $\underline{c}\ \underline{x}$) extreme point (via the pivot operation) until the optimal is reached. As there are roughly $N \equiv \binom{n}{m}$ such extreme points it would seem that this method might take many iterations but, surprisingly to some, this does not appear to be the case in practice.

In order to obtain a feel for whether or not the above is surprising, we consider a simple probabilistic (Markov chain) model as to how the algorithm moves along the extreme points. At first we suppose that if at any time the algorithm is at the jth best extreme point then after the next pivot the resulting extreme point is equally likely to be any of the $j - 1$ best. Under this assumption, we show that the time to get from the Nth best to the best extreme point has approximately, for large $N$, a Poisson distribution with mean equal to the logarithm (base $e$) of $N$. We also consider a more general probabilistic model

in which we drop the uniformity assumption and suppose that when at the jth best the next one is chosen probabilistically according to weights $w_i$ , $i = 1, \ldots, j - 1$ .

## 2. THE UNIFORM MARKOV CHAIN

Consider a Markov chain for which $P_{11} = 1$ and

$$P_{ij} = \frac{1}{i-1}, \; j = 1, \ldots, i-1, \; i > 1$$

and let $T_N$ denote the number of transitions to get from state $N$ to state $1$. Then $T_N$ can be expressed as

$$T_N = \sum_{j=1}^{N-1} I_j$$

where

$$I_j = \begin{cases} 1 & \text{if the process ever enters } j \\ 0 & \text{otherwise .} \end{cases}$$

Proposition 1:

$I_1, \ldots, I_{N-1}$ are independent and

$$P\{I_j = 1\} = 1/j \; , \; 1 \leq j \leq N-1 \; .$$

Proof:

Given $I_{j+1}, \ldots, I_N$ let $n = \min\{i : i > j \; , \; I_i = 1\}$. Then

$$P\{I_j = 1 \mid I_{j+1}, \ldots, I_N\} = \frac{1/(n-1)}{j/(n-1)} = 1/j \; . \blacksquare$$

Corollary 2:

(i) $E[T_N] = \sum_{j=1}^{N-1} 1/j$

(ii) $\mathrm{Var}\,(T_N) = \sum_{j=1}^{N-1} \frac{1}{j}\left(1 - \frac{1}{j}\right)$

(iii) For $N$ large, $T_N$ has approximately a Poisson distribution with mean $\log N$ .

Proof:

Parts (i) and (ii) follow from Proposition 1 and the representation $T_N = \sum_{j=1}^{N-1} I_j$ . Part (iii) follows from the Poisson limit theorem since

$$\int_1^N \frac{dx}{x} < \sum_1^{N-1} 1/j < 1 + \int_1^{N-1} \frac{dx}{x}$$

or

$$\log N < \sum_1^{N-1} 1/j < 1 + \log\,(N - 1)$$

and so

$$\log N \approx \sum_{j=1}^{N-1} 1/j \;. \blacksquare$$

## 3. APPLICATION TO SIMPLEX

Assuming that $n$, $m$ and $n - m$ are all large, we have by Stirling's approximation that

$$N = \binom{n}{m} \sim \frac{n^{n+1/2}}{(n-m)^{n-m+1/2} m^{m+1/2} \sqrt{2\pi}}$$

and so letting $c = n/m$

$$\log N \sim (mc + 1/2) \log (mc) - (m(c-1) + 1/2) \log (m(c-1)) - (m + 1/2) \log m - 1/2 \log (2\pi)$$

or

$$\log N \sim m\left[c \log \frac{c}{c-1} + \log (c-1)\right] .$$

Now, as $\lim_{x\to\infty} x \log (x/x-1) = 1$, it follows that when $c$ is large

$$\log N \sim m[1 + \log (c-1)] .$$

Thus for instance if $n = 8000$, $m = 1000$, then the number of necessary transitions is approximately Poisson distributed with mean $1000(1 + \log 7) \approx 3000$. As the variance is equal to the mean, we see by the normal approximation to the Poisson that the number of necessary transitions would be roughly between

$$3000 \pm 2\sqrt{3000} \text{ or , roughly, } 3000 \pm 110$$

95 percent of the time.

## 4. A WEIGHTED MARKOV CHAIN MODEL

Suppose now that $P_{11} = 1$ and

$$P_{ij} = \frac{w_j}{w_1 + \ldots + w_{i-1}} \qquad j \leq i - 1 .$$

With this model we are thus able to give more weight to those states closest to the one presently at by letting $w_j$ increase in $j$. Analogously with Proposition 1, we have

Proposition 2:

If

$$I_j = \begin{cases} 1 & \text{if } j \text{ is ever visited} \\ 0 & \text{otherwise} . \end{cases}$$

Then $I_1, \ldots, I_{N-1}$ are independent and

$$P\{I_j = 1\} = \frac{w_j}{\sum_{i=1}^{j} w_i} , \; 1 \leq j \leq N - 1 .$$

In addition, if $T_N = \sum_{j=1}^{N-1} I_j$. Then

$$E[T_N] = \sum_{j=1}^{N-1} \left( w_j / \sum_1^j w_i \right)$$

$$\mathrm{Var}\,(T_N) = \sum_{j=1}^{N-1} \frac{w_j}{\sum_1^j w_i} \left( 1 - \frac{w_j}{\sum_1^j w_i} \right) .$$

If for instance we use polynomial weights--$w_j = j^\alpha$ , $0 \le \alpha < \infty$ , then

$$\sum_{i=1}^{j} w_i = \sum_{i=1}^{j} i^\alpha$$

$$\approx \int_1^j x^\alpha dx$$

$$= \frac{j^{\alpha+1} - 1}{\alpha + 1}$$

and so

$$\frac{w_j}{\sum_1^j w_i} \approx \frac{(\alpha + 1)j^\alpha}{j^{\alpha+1} - 1} \approx \frac{\alpha + 1}{j} .$$

Hence

$$E[T_N] \approx \int_1^{N-1} \frac{\alpha + 1}{x} dx = (\alpha + 1) \log (N - 1)$$

and thus in this case $T_N$ has, for large $N$ , approximately a Poisson distribution with mean $(\alpha + 1) \log N$ . Thus when $N = \binom{n}{m}$ , the number of transitions (i.e., simplex pivot iterations) is approximately Poisson with mean

$$(\alpha + 1)m\left[c \log \left(\frac{c}{c - 1}\right) + \log (c - 1)\right] , \ c = n/m$$

which when $c$ is large is approximately

$$(\alpha + 1)m[1 + \log (c - 1)] .$$

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